Learning Outcome

Three-Phase Systems
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INTRODUCTION

This Workbook guides you through the learning outcomes related to:

Use of j Notation

GUIDANCE

This document is prepared to break the unit material down into bite size chunks. You will see the learning outcomes above treated in their own sections. Therein you will encounter the following structures;

**Purpose**

Explain why you need to study the current section of material. Quite often learners are put off by material which does not initially seem to be relevant to a topic or profession. Once you understand the importance of new learning or theory you will embrace the concepts more readily.

**Theory**

Conveys new material to you in a straightforward fashion. To support the treatments in this section you are strongly advised to follow the given hyperlinks, which may be useful documents or applications on the web.

**Example**

The examples/worked examples are presented in a knowledge-building order. Make sure you follow them all through. If you are feeling confident then you might like to treat an example as a question, in which case cover it up and have a go yourself. Many of the examples given resemble assignment questions which will come your way, so follow them through diligently.

**Question**

Questions should not be avoided if you are determined to learn. Please do take the time to tackle each of the given questions, in the order in which they are presented. The order is important, as further knowledge and confidence is built upon previous knowledge and confidence. As an Online Learner it is important that the answers to questions are immediately available to you. You will find the answers, upside down, below each set of questions. Contact your Unit Tutor if you need help.

**Challenge**

You can really cement your new knowledge by undertaking the challenges. A challenge could be to download software and perform an exercise. An alternative challenge might involve a practical activity or other form of research.

**Video**

Videos on the web can be very useful supplements to your distance learning efforts. Wherever an online video(s) will help you then it will be hyperlinked at the appropriate point.
1.1 Use of j Notation

When we have circuits containing just resistors then life is so easy in terms of circuit analysis. Most useful circuits also contain capacitors and inductors (usually coils and windings). The introduction of capacitors and inductors into circuits causes ‘phase angles’ in our calculations. The study of these phase angles is made much easier by introducing complex numbers.

From your level 3 studies you will have come across Inductive Reactance ($X_L$) and Capacitive Reactance ($X_C$). These terms are used to quantify the amount of ‘opposition’ caused by capacitors and inductors to changes in current or voltage. The term ‘reactance’ is brought about because capacitors cannot be charged or discharged in zero time, and inductors cannot be energised or de-energised in zero time. A good analogy for capacitors is the amount of water in a bathtub. It is impossible to fill a bathtub in zero time, and it’s also impossible to empty a bathtub in zero time. The amount of reactance from capacitors and inductors is a function of their manufactured properties and the frequency of operation. Let’s review the equations for these reactances...

\[
X_L = 2\pi fL \quad [\Omega] \\
X_C = \frac{1}{2\pi fC} \quad [\Omega]
\]

where;

- $X_L =$ inductive reactance (measured in Ohms, $\Omega$)
- $X_C =$ capacitive reactance (measured in Ohms, $\Omega$)
- $f =$ frequency (measured in Hertz, Hz)
- $L =$ inductance (measured in Henries, $H$)
- $C =$ capacitance (measured in Farads, $F$)

Consider the RLC circuit below...

![RLC Circuit Diagram]

We can draw a phasor diagram for this circuit, as follows...
The black arrow represents resistance. The current through a resistor is always in phase with the voltage across it. We place resistance on the horizontal axis.

The red arrow represents inductive reactance. We see that this leads the resistance by 90 degrees \((\pi/2 \text{ radians})\). We name this axis the ‘+j axis’. Mathematicians tend to designate this the ‘+i’ (imaginary) axis. Engineers do not use i since it clashes with the current symbol, so we use ‘j’ instead.

The blue arrow represents capacitive reactance. We see that this lags the resistance by 90 degrees \((\pi/2 \text{ radians})\). We designate this the ‘-j’ axis.

The dashed lines represent a graphical method of finding the resultant of these phasors, drawn in green. We term this resultant the \textit{impedance} of the circuit and mark it with ‘r’ for resultant. This resultant impedance makes an angle with the horizontal axis, marked with \(\phi\).

The resultant impedance is given the symbol \(Z\) for calculation purposes. We see that the green resultant has both horizontal and vertical components. The horizontal contribution is known as the \textit{real} component and the vertical contribution is known as the \textit{imaginary} component.

We may use Pythagoras’ theorem to denote impedance as follows...

\[
Z^2 = R^2 + X^2 \\
\therefore Z = \sqrt{R^2 + X^2} \quad [\Omega]
\]

In complex number notation we represent \(Z\) as...

\[
Z = R + j(X_L - X_C) \quad [\Omega]
\]
**Worked Example 1**

The series combination of a 100Ω resistor and a 10mH inductor form an impedance. If the circuit frequency is 10kHz determine the impedance of the circuit in complex number form.

We have the real part already. It is 100Ω (the value of the resistor). We calculate the imaginary part of the circuit as follows...

\[ X_L = 2\pi fL = 2\pi \times 10^4 \times 10 \times 10^{-3} = 628.3\Omega \]

We denote a complex number as follows...

**complex number = real part + j( imaginary part)**

So we may answer the question by writing...

\[ Z = (100 + j628.3)\ [\Omega] \]

**Worked Example 2**

If a complex number is given by \( Z = (100 + j628.3)\ [\Omega] \) find its Polar Form.

The Polar Form is given by the length of the resultant impedance (represented by \( r \) and the green phasor above) and the associated angle (\( \phi \)) of the resultant impedance, as follows...

\[ Z = r \angle \phi \ [\Omega] \]

Basic trigonometry tells us that...

\[ r = \sqrt{R^2 + X_L^2} = \sqrt{100^2 + 628.3^2} = 636.2\ \Omega \]

\[ \phi = \tan^{-1}\left(\frac{X_L}{R}\right) = \tan^{-1}\left(\frac{628.3}{100}\right) = 80.96^\circ \]

So the Polar Form is given by...

\[ Z = 636.2 \angle 80.96^\circ \ [\Omega] \]

Quite often we need to add, subtract, multiply and divide complex numbers. We may also like to multiply and divide Polar numbers. Let’s look at the mechanics of these operations.

We define \( j \) as the square root of -1...

\[ j = \sqrt{-1} \]
\[ j + j = 2j \]
\[ 5j - 3j = 2j \]
\[ j \times j = j^2 = \sqrt{-1} \times \sqrt{-1} = -1^{0.5} \times -1^{0.5} = -1^{0.5+0.5} = -1 \]
\[ \frac{6j}{3j} = 2 \]

**Worked Example 3**

A circuit current calculation involves the division of a voltage by an impedance...

\[ i = \frac{10 - j30}{3 + j4} \]

Determine the value of the current.

To perform such calculations we need to determine the complex conjugate of the denominator and then multiply this complex conjugate by both the numerator and denominator. The process is...

\[
\frac{\text{complex number } 1}{\text{complex number } 2} = \frac{\text{complex number } 1}{\text{complex number } 2} \times \frac{\text{complex conjugate of number } 2}{\text{complex conjugate of number } 2}
\]

The use of the complex conjugate actually simplifies our task because the new denominator becomes a purely real number.

*The complex conjugate of a complex number is simply the same complex number with the sign on the \( j \) term negated.*

So, we can write...

\[ i = \frac{10 - j30}{3 + j4} = \frac{(10 - j30)}{(3 + j4)} \times \frac{(3 - j4)}{(3 - j4)} \]

\[ = \frac{30 - j40 - j90 - j^2 120}{9 - j12 + j12 - j^2 16} \]

We know that \( j^2 = -1 \) so we may now say...

\[ = \frac{30 - j40 - j90 - (-1) \times 120}{9 - j12 + j12 + (-1) \times 16} = \frac{30 - j40 - j90 - 120}{9 - j12 + j12 + 16} = \frac{30 - j40 - j90 - 120}{25} \]

\[ = (-3.6 - j5.2) \ [A] \]

Such calculations are rather messy, as you can see. Fortunately, using the Polar Form of complex numbers when performing divisions leads to shorter calculations.

**Worked Example 4**
A circuit current calculation involves the division of a voltage by an impedance...

\[ i = \frac{12 - j10}{6 - j9} \]

Determine the value of the current using the Polar Form.

A quick conversion on a scientific calculator will transform our problem into...

\[ i = \frac{15.62\angle-39.8^\circ}{10.82\angle-56.31^\circ} [A] \]

When faced with such a Polar division the answer is determined in a quite straightforward manner...

**Polar Division: Divide the magnitudes, subtract the angles**...

\[ i = \frac{15.62\angle-39.8^\circ}{10.82\angle-56.31^\circ} \angle(-39.8 - (-56.31)) = 1.44\angle16.51^\circ [A] \]

Another common task is to multiply the Polar form of two complex numbers, as illustrated in the next worked example.

**Worked Example 5**

A circuit voltage is determined by multiplying current times impedance. The calculation is...

\[ V = 10\angle40^\circ \times 3\angle50^\circ \text{ volts} \]

Determine the voltage in Polar form.

When faced with such a Polar multiplication the answer is again determined in a quite straightforward manner...

**Polar Multiplication: Multiply the magnitudes, add the angles**...

\[ V = 10\angle40^\circ \times 3\angle50^\circ = 30\angle90^\circ \text{ volts} \]

1.2 Three-Phase Circuit Problems using j Notation

We are now armed with the mathematical tools needed to analyse three-phase circuits. Consider the unbalanced 4-wire star system below...
We notice that phase voltage 1 is entirely across phase impedance 1. We may therefore say...

\[ i_1 = \frac{V_1}{Z_1} = \frac{230 \angle 0^\circ}{115 \angle 10^\circ} = 2 \angle -10^\circ \ [A] \]

The situation is similar for the other two phases...

\[ i_2 = \frac{V_2}{Z_2} = \frac{230 \angle 120^\circ}{5 \angle -60^\circ} = 46 \angle 180^\circ \ [A] \]
\[ i_3 = \frac{V_3}{Z_3} = \frac{230 \angle 240^\circ}{10 \angle 40^\circ} = 23 \angle 200^\circ \ [A] \]

The sum of these three phase currents is equal to the neutral current \((I_N)\) in this star configuration. Unfortunately, we cannot add Polar quantities (division and multiplication are ok though, as we’ve seen) so we need to convert each of them into \(a + jb\) form. Some quick conversions on the calculator yield...

\[ i_1 = \frac{V_1}{Z_1} = \frac{230 \angle 0^\circ}{115 \angle 10^\circ} = 2 \angle -10^\circ \equiv 1.97 - j0.35 \ [A] \]
\[ i_2 = \frac{V_2}{Z_2} = \frac{230 \angle 120^\circ}{5 \angle -60^\circ} = 46 \angle 180^\circ \equiv -46 + j0 \ [A] \]
\[ i_3 = \frac{V_3}{Z_3} = \frac{230 \angle 240^\circ}{10 \angle 40^\circ} = 23 \angle 200^\circ \equiv -21.61 - j7.87 \ [A] \]

Adding the complex numbers gives...

\[ i_N = (1.97 - 46 - 21.61) + j(-0.35 + 0 - 7.87) = -65.64 - j8.22 \equiv 66.15 \angle -172.86^\circ \ [A] \]

Let us now consider the voltage between two separate phases. Look at the phasor diagram below...
The diagram on the left illustrates phase 1 in red ($V_{P1}$). For the sake of analytical simplicity it is given a magnitude of 1 volt, although this could be scaled up to any voltage you like. The solid blue phasor represents phase 2, which is 120 degrees out of phase with phase 1. To work out the difference in voltage between these two phases we must find invert phase 2, giving $-V_{P2}$, shown dashed in blue. Our task is to find the resultant of $V_{P1}$ and $-V_{P2}$. This task is performed in the diagram on the right.

The green phasor represents the resultant line voltage, $V_L$. This is formed by drawing a parallelogram based upon $V_{P1}$ and $-V_{P2}$. To work out the magnitude of the line voltage in green we apply Pythagoras’ theorem...

$$|V_L| = \sqrt{(1 + 0.5)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$|V_L| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{\frac{12}{4}} = \sqrt{3} \text{ volts}$$

This then proves that the line voltage (from phase to phase) is $\sqrt{3}$ times the phase voltage. Therefore, if we have a phase voltage of 230 V then the line voltage will be $\sqrt{3} \times 230 = 398.37$ volts. This figure tends to be rounded to 400 V in the UK since it is impossible to maintain an exact voltage on the distribution system.

In the UK the consumer supply voltage is 230 V +10%/-6%. This means that the phase voltage can rise to 253V and fall to 216.2 V. If we look at the line voltage then it can be as high as 438.2 volts and as low as
374.5 volts. The figure of 438.2 volts for maximum line voltage tends to be rounded to 440 volts for normal everyday use and signage.

### Worked Example 6

Three identical coils, each of resistance $20\,\Omega$ and inductance $200\,mHz$, are connected in a Delta configuration to a 230 volt, 50Hz 3-phase supply. Determine the magnitude of each load current.

The first step here is to determine the load impedance on each phase...

$$X_L = \frac{2\pi f L}{\Omega}$$

$$= \frac{2\pi \times 50 \times 0.2}{20\,\Omega} = 62.83\,\Omega$$

$$\therefore Z_L = 20 + j62.83 \equiv 65.94\angle72.34^\circ \,[\Omega]$$

The circuit is shown below.

![Delta Configuration Circuit](image)

We already know that the line voltage magnitude is $\sqrt{3}$ times the phase voltage...

$$Line \ voltage \ magnitude = \sqrt{3} \times 230 = 398.37\,V$$

We also notice that each load impedance has a line voltage connected. We simply need to divide our magnitudes for voltage and impedance to find the magnitude of each load current...

$$Magnitude \ of \ each \ load \ current = \frac{398.37}{65.94} = 6.04\,A$$

There are two common ways to measure the total effective power dissipated in the loads. The two-wattmeter method is employed for balanced loads. The three-wattmeter method is employed for balanced or unbalanced loads.
Worked Example 7

Draw a star-star 230V, 50Hz, 3-phase system with BALANCED loads of 20∠30° [Ω] on the TINA-TI simulator. Use both the two-wattmeter and three-wattmeter methods to determine the total effective power dissipated in the load system.

Useful starter video on TINA-TI

The TINA-TI free simulator is available [here](#). Before we draw the two measurement circuits we need to know how to represent a load impedance of 20∠30° [Ω] at 50Hz.

Convert the Polar form of the given impedance into its complex number form using a calculator...

\[ 20∠30° \equiv 17.32 + j10 \]

We see that the j part is positive so we are looking for an inductor. To determine the value of the inductor we proceed as follows...

\[ X_L = 2\pi fL = 10 \quad \therefore L = \frac{X_L}{2\pi f} = \frac{10}{2\pi \times 50} = 0.032 \, H \]

So each load consists of a resistor of 17.32Ω and a series inductor of 0.032 H. We then place these into our star load and construct the measurement circuits. The simulation is started by clicking **ANALYSIS->AC ANALYSIS->CALCULATE NODAL VOLTAGES...**
The total power is found in each case by adding the readings on the wattmeters. The two-wattmeter measurement produces $1.14\text{kW} + 2.29\text{kW} = 3.43\text{kW}$. The three-wattmeter method produces three lots of $1.14\text{kW} = 3.42\text{kW}$. There is only a minor difference here, caused by our truncation whilst calculating the inductance value.

**Challenge**

Reproduce the above results on the TINA-TI simulator.

**Worked Example 8**

We turn now to the situation where there are faults in our three-phase systems. Consider the system below which has the first phase open-circuited...
Here we can see that the neutral current will comprise of $i_2 + i_3$. Let’s do the calculation...

$$i_N = i_2 + i_3 = \frac{V_2}{Z_2} + \frac{V_3}{Z_3} = \frac{230\angle120^\circ}{5\angle-60^\circ} + \frac{230\angle240^\circ}{10\angle40^\circ} = 46\angle180^\circ + 23\angle200^\circ$$

Since we cannot directly add Polar numbers we must convert to complex number form, do the addition, then convert back to Polar form...

$$(-46 + j0) + (-21.61 - j7.87) = -67.61 - j7.87 \equiv 68.1\angle-173.4^\circ$$ [A]

*Note: When you see a load impedance with a negative angle then the reactive component involved is a capacitance rather than an inductance. This is handy to know when you are checking your calculations on the TINA-TI simulator.*

**Challenge**

Reproduce the above results on the TINA-TI simulator.

**Worked Example 9**

What about the situation where we have impedance 1 developing an open circuit? In this case we would like to determine the current flowing in impedance 2. Let’s look at the scenario again...
The current flowing in $Z_2$ would simply be given by $V_2/Z_2$. We may therefore easily calculate...

$$i_2 = \frac{V_2}{Z_2} = \frac{230\angle120^\circ}{5\angle-60^\circ} = 46\angle180^\circ = -46 \, \text{A}$$

When checking these calculations on the simulator you may get a negative sign for current, rather than an expected positive quantity. It depends which way around you have placed your supply voltage. Try it and see.

**Challenge**

Reproduce the above results on the TINA-TI simulator.