Unit 5: Electrical & Electronic Principles

Unit Workbook 4

in a series of 4 for this unit

Learning Outcome:

RLC Transients
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INTRODUCTION

This Workbook guides you through the learning outcomes related to:

**Laplace transforms**: definition of the Laplace transform of a function; use of a table of Laplace transforms

**Transient analysis**: expressions for component and circuit impedance in the s-plane; first order systems must be solved by Laplace (i.e. RL and RC networks); second order systems could be solved by Laplace or computer-based packages

**Circuit responses**: over, under, zero and critically damped response following a step input; zero initial conditions being assumed

GUIDANCE

This document is prepared to break the unit material down into bite size chunks. You will see the learning outcomes above treated in their own sections. Therein you will encounter the following structures;

- **Purpose**: Explains why you need to study the current section of material. Quite often learners are put off by material which does not initially seem to be relevant to a topic or profession. Once you understand the importance of new learning or theory you will embrace the concepts more readily.

- **Theory**: Conveys new material to you in a straightforward fashion. To support the treatments in this section you are strongly advised to follow the given hyperlinks, which may be useful documents or applications on the web.

- **Example**: The examples/worked examples are presented in a knowledge-building order. Make sure you follow them all through. If you are feeling confident then you might like to treat an example as a question, in which case cover it up and have a go yourself. Many of the examples given resemble assignment questions which will come your way, so follow them through diligently.

- **Question**: Questions should not be avoided if you are determined to learn. Please do take the time to tackle each of the given questions, in the order in which they are presented. The order is important, as further knowledge and confidence is built upon previous knowledge and confidence. As an Online Learner it is important that the answers to questions are immediately available to you. You will find the answers, upside down, below each set of questions. Contact your Unit Tutor if you need help.

- **Challenge**: You can really cement your new knowledge by undertaking the challenges. A challenge could be to download software and perform an exercise. An alternative challenge might involve a practical activity or other form of research.

- **Video**: Videos on the web can be very useful supplements to your distance learning efforts. Wherever an online video(s) will help you then it will be hyperlinked at the appropriate point.
4.1 Laplace Transforms

4.1.1 Definition of the Laplace Transform of a Function
If the independent variable of a function is time (t) then the function may be written as \( f(t) \). The Laplace Transform of \( f(t) \) is given by...

\[
\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt
\]

Here, s is a complex number in the frequency domain. We can think of s as merely a parameter to manipulate. We may represent it as \( s \equiv j\omega \), and this equivalence will be useful to us later on.

When we analyse circuits containing combinations of resistors, capacitors and/or inductors it is quite usual to come up with first and second-order differential equations. To ease this burden of analysis we use Laplace Transforms, which convert circuit analysis into a purely algebraic problem.

Laplace Transforms may be deduced by performing the integration, as above, but this soon becomes a very tedious exercise. Virtually all of the Laplace Transforms we meet in engineering have already been evaluated (i.e. the integrations are done for us). We simply need to use a table containing these evaluated transforms.

It is informative at this point to try a couple of these integrations for a unit step source (closing a switch to connect to a DC supply) and a unit ramp source (linearly increasing to 1 V DC source). These exemplify the reason why we wish to avoid the additional mathematics.

**Find the Laplace Transform of \( f(t) = 1 \) (unit step function)**

\[
\mathcal{L}\{1\} = \int_0^\infty e^{-st}(1) \, dt = \left[ \frac{e^{-st}}{-s} \right]_0^\infty = \frac{1}{-s} (e^{-\infty} - e^{-0}) = \frac{0 - 1}{-s} = \frac{-1}{-s} = \frac{1}{s}
\]

That function was just the number 1, so any number can be evaluated in the same way i.e. \( \mathcal{L}\{8\} = \frac{8}{s} \).

**Find the Laplace Transform of \( f(t) = t \) (unit ramp function)**

\[
\mathcal{L}\{t\} = \int_0^\infty e^{-st}(t) \, dt
\]

Integration by parts is required here, as it is for virtually all functions for which we need to find the Laplace Transform. Let’s do that then...
\[ \mathcal{L}\{t\} = \int_0^\infty e^{-st} \, dt = \left[ \frac{te^{-st}}{-s} - \int_0^\infty \frac{e^{-st}}{-s} \, dt \right]_0^\infty = \left[ \frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_0^\infty = \left[ \frac{-s e^{-st}}{s^2} - 0 \right] - \left[ \frac{-s}{s^2} - \frac{1}{s^2} \right] = \frac{1}{s^2} \]

We have managed to find the right answer to these small problems. You could perform all the necessary integrations for many other functions, but why would you need to do that when they have already been done for you? The results are found in a table of Laplace Transforms, as previously mentioned, which we now turn to.

### 4.1.2 Using a Table of Laplace Transforms

Here is a short table of Laplace Transforms which will be very useful when analysing circuits. Have a look through it (two of them you already know). We shall then discuss how it is used.

<table>
<thead>
<tr>
<th>Function of time, ( f(t) )</th>
<th>Laplace Transform of ( f(t) ), ( \mathcal{L}{f(t)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ( \delta ) (unit impulse)</td>
<td>1</td>
</tr>
<tr>
<td>2 ( 1 ) (unit step function)</td>
<td>( \frac{1}{s} )</td>
</tr>
<tr>
<td>3 ( t ) (unit ramp)</td>
<td>( \frac{1}{s^2} )</td>
</tr>
<tr>
<td>4 ( e^{at} ) (exponential growth)</td>
<td>( \frac{1}{s-a} )</td>
</tr>
<tr>
<td>5 ( e^{-at} ) (exponential decay)</td>
<td>( \frac{1}{s+a} )</td>
</tr>
<tr>
<td>6 ( \sin(\omega t) )</td>
<td>( \frac{\omega}{s^2 + \omega^2} )</td>
</tr>
<tr>
<td>7 ( \cos(\omega t) )</td>
<td>( \frac{s}{s^2 + \omega^2} )</td>
</tr>
<tr>
<td>8 ( e^{-at}\sin(\omega t) ) (decaying sine wave)</td>
<td>( \frac{\omega}{(s+a)^2 + \omega^2} )</td>
</tr>
<tr>
<td>9 ( e^{-at}\cos(\omega t) ) (decaying cosine wave)</td>
<td>( \frac{s+a}{(s+a)^2 + \omega^2} )</td>
</tr>
<tr>
<td>10 ( \frac{d f(t)}{dt} ) (first differential)</td>
<td>( s F(s) - f(0) )</td>
</tr>
<tr>
<td>11 ( \frac{d^2 f(t)}{dt^2} ) (second differential)</td>
<td>( s^2 F(s) - sf(0) - \frac{d f(0)}{dt} )</td>
</tr>
<tr>
<td>12 ( \int f(t) , dt ) (integral)</td>
<td>( \frac{1}{s} F(s) + \frac{1}{s} f(0) )</td>
</tr>
</tbody>
</table>
We have already mentioned the fact that Laplace Transforms allow us to convert differential equations into easy algebraic problems – that’s their beauty. Here is a procedure for analysing a circuit and using the table to find solutions to circuit parameters.

1) Represent the circuit in terms of an expression involving voltage, current and component values
2) Transform each term in the expression separately, using the table
3) Simplify the expression as best as you can. If there are any initial values then put them in. Enter the component values given in the circuit. Ensure that the simplifications represent expressions which the table can handle.
4) Having found an expression which looks like one of the terms on the right-hand-side of the table, look to the left to find the inverse Laplace Transform (i.e. the circuit solution).

This whole topic and process never looks too understandable the first time you meet it. Soon enough you shall begin to see the light by way of example circuits.

4.2 Transient Analysis

4.2.1 Expressions for Component and Circuit Impedance in the s-plane

It was mentioned earlier that we can represent $s$ as $j\omega$. When we analyse the reactance of an inductor we have the usual expression...

$$+jX_L = +j2\pi f L = +j\omega L$$

Notice the $j\omega$ in that expression. We are saying that $j\omega$ can be represented by $s$, so let’s do that...

$$+j\omega L \equiv sL$$

When we analyse the reactance of a capacitor we have the usual expression...

$$-jX_C = -j \times \frac{1}{2\pi f C} = -j \times \frac{1}{\omega C} = \frac{1}{j\omega C}$$

Notice the $j\omega$ in that expression also. We are saying that $j\omega$ can be represented by $s$, so let’s do that...

$$-jX_C \equiv \frac{1}{sC}$$

We know that inductive and capacitive reactance are dependent upon the frequency used. Their values will vary with frequency. Pure resistors are different, their values do not vary with frequency, so we just think of resistors as constants in our expressions. If we had a resistor and inductor in series then we can evaluate the overall impedance of the combination in the s-domain as follows...
\\( Z(s) = R + sL \)

If we had a resistor and capacitor in series then we can evaluate the overall impedance of the combination in the s-domain as follows...

\\( Z(s) = R + \frac{1}{sC} \)

If we had a resistor and inductor in parallel then we can evaluate the overall impedance of the combination in the s-domain as follows...

\\( Z(s) = \frac{R \times sL}{R + sL} = \frac{sLR}{R + sL} \)

If we had a resistor and capacitor in parallel then we can evaluate the overall impedance of the combination in the s-domain as follows...

\\( Z(s) = \frac{R \times \frac{1}{sC}}{R + \frac{1}{sC}} = \frac{R}{sCR + 1} \)

If we had a resistor, inductor and capacitor all in series then we can evaluate the overall impedance of the combination in the s-domain as follows...

\\( Z(s) = R + sL + \frac{1}{sC} \)

If we had a resistor, inductor and capacitor all in parallel then we can evaluate the overall impedance of the combination in the s-domain as follows...

\\( \frac{1}{Z(s)} = \frac{1}{R} + \frac{1}{sL} + \frac{1}{sC} = \frac{1}{R} + \frac{1}{sL} + sC \)

After finding the common denominator, manipulating and flipping the expression we get...

\[ Z(s) = \frac{sLR}{s^2RLC + sL + R} \]

It doesn’t matter how complicated the circuit is. There may be series elements and parallel elements. Just analyse them as you see them, then write down the expression in the s-domain.

### 4.2.2 Laplace Solution of First Order Systems

We finally come to the real nitty-gritty of all this Laplace theory – applying it to real circuits to see if it can shorten our solutions by not having to solve differential equations. Let’s look at a couple of worked examples.
Worked Example 1

Considering the circuit below, use Laplace Transforms to find an expression for the current flowing after the switch is closed. Assume \( i = 0 \) when \( t = 0 \).

![Circuit Diagram]

We start our solution by using Kirchhoff’s Voltage Law (KVL) around the circuit, once the switch is closed...

\[ E = V_R + V_L \]

We then form a first order differential equation involving the current in the circuit. The current is a function of time so we shall denote it by \( i(t) \)...

\[ E = Ri(t) + L \frac{di(t)}{dt} \]

Our next step is to take Laplace Transforms (LT’s) for the terms in the expression we currently have. Our table of Laplace Transforms now becomes useful.

\( E \) is a unit step function, since the battery voltage is introduced in a sudden ‘step’ by the closing of the switch. We can then write...

\[ \frac{E}{s} = Ri(s) + L[sF(s) - f(0)] \]

Some points to note here:

- When taking the LT of \( i(t) \) we simply write \( i(s) \)
- \( F(s) \) is our function for current, so we simply replace it with \( i(s) \)
- \( f(0) \) is the current when time is zero. This was stated as zero in the question, so we omit it.

\[ \therefore \frac{E}{s} = Ri(s) + Ls \cdot i(s) \]

\[ \therefore \frac{E}{s} = i(s)[R + Ls] \]

\[ \therefore i(s) = \frac{E}{s(R + Ls)} \]

We are trying to get this last expression into a format whereby we can read off the answer in the table from right to left. It doesn’t look like anything in the table at the moment. What we must do is to take partial fractions (remember those from your Analytical Methods unit)....
\[
\frac{E}{s(R + Ls)} \equiv \frac{A}{s} + \frac{B}{R + Ls}
\]

Multiplying both sides by \(s(R + Ls)\) gives...

\[
E = A(R + Ls) + Bs
\]

We now need to find values for \(A\) and \(B\). The simplest starting point is to let \(s\) be zero...

**Let \(s = 0\)**

\[
\therefore E = AR
\]

\[
\therefore A = \frac{E}{R}
\]

Now that we have found \(A\) we must think of an easy way to find \(B\). If we make the brackets equal to zero would isolate \(B\) for us. In order to make the brackets equal to zero we must make \(s\) equal to \(-\frac{R}{L}\).

**Let \(s = -\frac{R}{L}\)**

\[
\therefore E = B \left(-\frac{R}{L}\right)
\]

\[
\therefore B = -\frac{EL}{R}
\]

We may now put these expressions for \(A\) and \(B\) into our main developed expression...

\[
\frac{E}{s(R + Ls)} \equiv \frac{A}{s} + \frac{B}{R + Ls} = \frac{E}{s} + \frac{-EL}{R} = i(s)
\]

\[
\therefore i(s) = \frac{E}{s} + \frac{-EL}{R + Ls}
\]

We now try to resolve this expression into a format which is table-friendly...

\[
i(s) = \frac{E}{R} \left(\frac{1}{s} - \frac{L}{Ls + R}\right)
\]

\[
i(s) = \frac{E}{R} \left(\frac{1}{s} - \frac{L}{L} s + \frac{R}{L}\right)
\]

\[
\therefore i(s) = \frac{E}{R} \left(\frac{1}{s} - \frac{1}{s + \frac{R}{L}}\right)
\]

There are two terms in the big brackets here. Both of them look like entries in our table of LTs. The first term is from row 2 of the table and the second term is from row 5 of the table. Notice that \(R/L\) is taken to
be the quantity 'a' mentioned in the table. We may now take inverse Laplace Transforms (usually written as $\mathcal{L}^{-1}\{F(s)\}$)...

$$i(t) = \frac{E}{R} \left(1 - e^{-\frac{Rt}{L}}\right)$$

We have managed to find an LT solution for our first circuit. We were given component values in the question so let’s put them into our new expression...

$$i(t) = \frac{10}{1} \left(1 - e^{-\frac{10t}{0.001}}\right)$$

$$\therefore i(t) = 10 \left(1 - e^{-1000t}\right) \ [A]$$

So much for an RL circuit then. Let’s now have a look at an RC circuit.

**Worked Example 2**

Considering the circuit below, use Laplace Transforms to find an expression for the current flowing after the switch is closed. Assume $i = 0$ when $t = 0$.

![Circuit Diagram]

We start our solution by using Kirchhoff’s Voltage Law (KVL) around the circuit, once the switch is closed...

$$E = V_R + V_C$$

We then form a first order differential equation involving the current in the circuit...

$$E = Ri(t) + \frac{1}{C} \int i(t) \, dt$$

Our next step is to take Laplace Transforms (LT’s) for the terms in the expression we currently have...

$$\frac{E}{s} = Ri(s) + \frac{1}{C} \left(\frac{1}{s} F(s) + \frac{1}{s} f(0)\right)$$

The initial conditions given are: $i = 0$ when $t = 0$, so we ignore $f(0)$. As before, we take $F(s)$ to be $i(s)$...
\[
\frac{E}{s} = Ri(s) + \frac{i(s)}{sC}
\]
\[
\therefore \frac{E}{s} = \frac{i(s)}{s} \left( R + \frac{1}{sC} \right)
\]
\[
\therefore i(s) = \frac{E}{s} \left( R + \frac{1}{sC} \right)
\]
\[
\therefore i(s) = \frac{E}{sR + \frac{1}{C}} = \frac{EC}{sRC + 1} = \frac{EC/RC}{s(\frac{1}{RC}) + (\frac{1}{RC})} = \frac{E/R}{s + (\frac{1}{RC})}
\]
\[
\therefore i(s) = \frac{E}{R} \left( \frac{1}{s + (\frac{1}{RC})} \right)
\]

We now need to take the inverse LT for the expression in the large brackets. Notice that 1/RC is the same quantity meant by ‘a’ in our table. Our answer for this circuit is therefore, again, taken from row 5 of the table...

\[
i(t) = \frac{E}{R} e^{-\frac{1}{RC}t}
\]

We were given component values in the question so let’s put them into our expression for current...

\[
i(t) = \frac{10}{1 \times 10^6} e^{-\frac{1}{1 \times 10^6 \times 1 \times 10^{-6}}t} = 10^{-5} e^{-t} \quad [A]
\]
4.2.3 Second Order Systems

First order systems are those where the highest differential is of order 1 – for example, they feature $\text{d}i/\text{d}t$ as the highest differential. You saw an example of this in the RL circuit analysed previously.

Second order systems feature a differential of order 2 i.e. $d^2i/dt^2$. These systems feature an inductor and a capacitor. Let’s look at a worked example involving a series RLC combination.

**Worked Example 3**

Considering the circuit below, use Laplace Transforms to find an expression for the current flowing after the switch is closed. Assume $i = 0$ when $t = 0$.

![Circuit Diagram]

After the switch is closed we can apply Kirchhoff’s Voltage Law...

$$E_1 = v_R + v_L + v_C$$

Take Laplace Transforms...

$$\frac{E_1}{s} = R_1 i(s) + L_1 s i(s) + \frac{i(s)}{sC}$$

At this point it is helpful to put in the component values (simplifies the algebra)...

$$\frac{20}{s} = 10 i(s) + 0.1 s i(s) + \frac{i(s)}{10^{-5} s}$$

$$\therefore \frac{20}{s} = i(s) \left( 10 + 0.1s + \frac{1}{10^{-5} s} \right)$$

$$\therefore i(s) = \frac{20}{s \left( 10 + 0.1s + \frac{1}{10^{-5} s} \right)} = \frac{20}{0.1s^2 + 10s + 10^5}$$

Now divide top and bottom by 0.1

$$\therefore i(s) = \frac{200}{s^2 + 100s + 10^6}$$

We now need to get the expression into a form suitable for table use. We wish to express it in the following form...
\[
\frac{\omega}{(s+a)^2 + \omega^2}
\]

... which is from row 8 of the table. The way to do this is to look at that 100 in the denominator, divide it by 2, which gives us 50, then square that 50 to give 2,500 which we then add \textit{and} subtract to the denominator. This process gives us...

\[
i(s) = \frac{200}{s^2 + 100s + 10^6} = \frac{200}{s^2 + 100s + 2500 - 2500 + 10^6}
\]

Adding and subtracting 2500 does not change it, of course. The value in doing this, however, is that we may factorise the first three terms in the denominator, as follows...

\[
s^2 + 100s + 2500 \equiv (s + 50)^2
\]

...which looks a bit more like the expression in row 8 of the table. The last two terms evaluate to...

\[-2500 + 1,000,000 = 997,500
\]

We need an \(\omega^2\) to represent 997,500, so the value of \(\omega\) must be the square root of 997,500 i.e.

\[
\omega^2 = 997,500 \quad : \quad \omega = \sqrt{997,500} \approx 998.75
\]

We can now bring these developments together...

\[
i(s) = \frac{200}{(s + 50)^2 + (998.75)^2}
\]

Now the denominator look very much like what we need. We still have a bit of work to do on the numerator though. We’d like \(\omega\) to appear there. The value of \(\omega\) is 998.75 but the numerator must evaluate to 200. We work out the numerator as follows...

\[
998.75 \times \text{some number} = 200
\]

\[
\therefore \text{some number} = \frac{200}{998.75} \approx 0.2
\]

We may now write...

\[
i(s) = \frac{0.2 \times 998.75}{(s + 50)^2 + (998.75)^2}
\]

Which can be written as...

\[
i(s) = 0.2 \left( \frac{998.75}{(s + 50)^2 + (998.75)^2} \right)
\]

The expression in the large brackets is now ready for an Inverse Laplace Transform (row 8 of the table remember)... 

\[
i(t) = 0.2e^{-50t} \sin(998.75 \ t) \quad [A]
\]
These calculations are fairly complex, so it is always valuable to use the MicroCap simulator (perform a Transient Analysis and make sure you turn OFF the 'Operating Point' checkbox). The simulator produces the following waveform for this circuit...

Notice that waveform has a decaying exponential envelope. The yellow marker is there so that a check can be made on our calculations. The simulator is saying that after 20 ms the circuit current has a value of around 66 mA. Let’s use our derived expression to see if this is so...

$$i(t) = 0.2e^{-50t} \sin(998.75 \times t) \quad [A]$$

Let $t = 0.02$

$$i(0.02) = 0.2e^{-50 \times 0.02} \sin(998.75 \times 0.02) = 66.4 \text{ mA}$$

Looks like we’re ok! If you zoom right in on the simulator result at 20 ms the answer will be exactly right (it’s just a matter of mathematical resolution).
4.3 Circuit Responses

4.3.1 Theory

For a series RLC circuit there are number of ways that the circuit response may behave. We explore these behaviours here with the initial aid of some revision from the Further Analytical Methods unit – Ordinary Differential Equations (worth a quick revision of Workbook 4 for that unit).

Given a series RLC circuit connected to some supply voltage $V$ we may employ Kirchhoff’s Voltage Law and write...

$$V = v_R + v_L + v_C \quad [1]$$

We know that...

$$v_L = L \frac{di}{dt} \quad [2]$$

We also know that...

$$i = C \frac{dv_C}{dt} \quad [3]$$

If we substitute [3] into [2] we get...

$$v_L = L \frac{d}{dt} \left\{ C \frac{dv_C}{dt} \right\} = LC \frac{d^2v_C}{dt^2} \quad [4]$$

Furthermore, we know...

$$v_R = iR = \left( C \frac{dv_C}{dt} \right) R$$

$$v_R = RC \frac{dv_C}{dt} \quad [5]$$


$$V = RC \frac{dv_C}{dt} + LC \frac{d^2v_C}{dt^2} + v_C$$

Rearranging...

$$LC \frac{d^2v_C}{dt^2} + RC \frac{dv_C}{dt} + v_C = V \quad [6]$$
Equation [6] is a linear 2\textsuperscript{nd} order differential equation. To solve it we turn it into a homogeneous type by forcing V to zero...

$$LC \frac{d^2v_c}{dt^2} + RC \frac{dv_c}{dt} + v_c = 0 \quad [7]$$

The standard solution here is...

Let  $v_c = Ae^{mt} \quad [8]$  

∴  $\frac{dv_c}{dt} = Am e^{mt} \quad [9]$  

∴  $\frac{d^2v_c}{dt^2} = Am^2e^{mt} \quad [10]$  

Substituting [8], [9] and [10] into [7] gives...

$$LC (Am^2e^{mt}) + RC (Am e^{mt}) + Ae^{mt} = 0$$

Taking out the common factor...

$$Ae^{mt} (m^2LC + mRC + 1) = 0 \quad [11]$$

We can then say that $Ae^{mt}$ is a solution only if...

$$m^2LC + mRC + 1 = 0 \quad [12]$$

Equation [12] is known as the **Auxiliary Equation** (remember?)  

Our auxiliary equation is in quadratic form, which we must solve using the quadratic formula...

$$m = \frac{-RC \pm \sqrt{(RC)^2 - 4(LC)(1)}}{2LC} = \frac{-RC \pm \sqrt{R^2C^2 - 4LC}}{2LC}$$

$$m = \frac{-RC}{2LC} \pm \sqrt{\frac{R^2C^2 - 4LC}{(2LC)^2}} = \frac{-R}{2L} \pm \sqrt{\frac{R^2C^2}{4L^2C^2} - \frac{4LC}{4L^2C^2}}$$

∴  $m = \frac{-R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \quad [13]$  

We shall next investigate the possible properties of equation [13] to determine the characteristics of an RLC series circuit.
4.3.2 Overdamped Response following a Step Input

\[ m = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \]

If \( m \) has two different real roots then the series RLC circuit is said to be **overdamped**. For this to be the case then...

\[ \left(\frac{R}{2L}\right)^2 > \frac{1}{LC} \]

An overdamped series RLC circuit is shown below, along with its transient response...

The transient current dies away very slowly with time.
4.3.3 Underdamped Response following a Step Input

\[ m = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \]

If \( m \) has two complex roots then the series RLC circuit is said to be underdamped. For this to be the case then...

\[ \left(\frac{R}{2L}\right)^2 < \frac{1}{LC} \]

An underdamped series RLC circuit is shown below, along with its transient response...

The current oscillates around a steady value and dies away slowly.
### 4.3.4 Critically-damped Response following a Step Input

$$m = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

If $m$ has **two real equal roots** then the series RLC circuit is said to be **critically-underdamped**. For this to be the case then...

$$\left(\frac{R}{2L}\right)^2 = \frac{1}{LC}$$

A critically-damped series RLC circuit is shown below, along with its transient response...

Notice that this transient response looks similar to the overdamped case. The difference is that the response dies away much more quickly in the critically-damped circuit.
4.3.5 Zero-damped Response following a Step Input

In this case the resistance is zero. The circuit is also known as being ‘undamped’. A zero-damped series RLC circuit is shown below, along with its transient response...

Assuming an ideal source (no internal resistance) then the oscillations continue indefinitely.